S.-T. Yau College Student Mathematics Contests 2023

Oral Exams in Geometry and Topology

Individual (Solve 3 out of 4 problems)

1. Show that $S^2 \times S^1$ is a double cover of the connected sum $\mathbb{R}P^3 \# \mathbb{R}P^3$.

2. Consider the matrix group

$$G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : \ a, b, c \in \mathbb{R} \right\}.$$

Prove that its exponential map from its Lie algebra

$$\exp : \mathfrak{G} \to G$$

is a diffeomorphism. Provide a generalization and justification of the statement.

3. In the four-dimensional Euclidean space, consider the smooth hypersurface H

$$\{(x, y, z, w) \mid x^2 + 2y^2 + 3z^2 + 4w^2 = 1\}.$$

Does there exist a 3-dimensional linear subspace which intersects H along a round sphere? Explain.

4. Let (M, ds^2) be a closed surface. Prove that the first nonzero eigenvalue $\lambda_1(M)$ of the Laplacian satisfies

$$\lambda_1(M) \cdot \mathrm{Vol}(M) \leqslant 2V_c(n, M)$$

for all n where the n-conformal volume $V_c(n, M)$ of M is defined. Equality implies that M must be a minimal surface of the unit n-sphere $\mathbb{S}^n$ and the immersion is given by a subspace of the first eigenspace. Here $\mathrm{Vol}(M)$ denotes the volume of M.

Hint of Problem 4: Let $\phi : M^m \to \mathbb{S}^n$ be a conformal map from a m-dimensional compact Riemannian manifold into the unit n-sphere $(\mathbb{S}^n, ds_0^2)$. Let G denote the group of conformal diffeomorphisms of $\mathbb{S}^n$. Define

$$V_c(n, \phi) = \sup_{g \in G} \int_M d\mu_g = \sup_{g \in G} \int_M |\nabla(g \circ \phi)|^2 d\mu_M,$$

where $d\mu_g$ is the volume element (possibly degenerate) associated to the tensor $\phi^*(g^* ds_0^2)$.

For any compact Riemannian manifold M, its *n-conformal volume* is defined to be

$$V_c(n, M) = \inf_{\phi} V_c(n, \phi),$$

where ϕ runs over all non-degenerate conformal mappings of M into $\mathbb{S}^n$.